MAGNETIC FIELD IN A CYLINDRICAL CONDUCTOR MOVING WITH A VELOCITY PROPORTIONAL TO r^{-1}

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UDC 538.4

Nonstationary magnetic field problems in a moving conductor are of interest in connection with obtaining pulsed magnetic fields by magnetic cumulation [1]. The field penetrates into the conductor as a result of the growth of the skin layer and is carried along with the conductor. The first mechanism of the interaction of a field with a conductor is called the diffusion of the field, and the second convection. Five self-similar solutions of magnetic field problems in a conductor which has a velocity $v = q/2\pi r$ and a conductivity σ =const are discussed and a numerical solution of the problem of the compression of a field in a cylindrical cavity when the conductor moves toward the axis is presented. One of the self-similar solutions is compared with the numerical solution.

1. In the problems under discussion the magnetic field in a conductor is described by the solution of the equation

$$\frac{c^2}{4\pi\sigma}\frac{\partial^2 B}{\partial r^2} + \left(\frac{c^2}{2\sigma} - q\right)\frac{1}{2\pi r}\frac{\partial B}{\partial r} - \frac{\partial B}{\partial t} = 0 \quad (1.1)$$

which satisfies the condition for the continuity of the field at the boundary of the conductor $r = r_*(t)$

$$B|_{r=r_*} = B_*(t) \quad (1.2)$$

and the condition for the diffusion of the flux

$$\frac{d\Phi_*}{dt} = \frac{c^2}{2\sigma}\left(r\frac{\partial B}{\partial r}\right)_{r=r_*} \quad (1.3)$$

which follows from Ohm's law and the equations of the quasistationary electromagnetic field. The field $B_*(t)$ at the boundary of the conductor is assumed uniform but can depend on time, and $\Phi_*(t)$ is the flux in a cylinder of radius $r_*(t)$.

Motion of the conductor away from the axis corresponds to $q > 0$, and motion toward the axis to $q < 0$. For a steady source strength the boundary of the conductor is given by

$$r_* = (qt\pi^{-1})^{1/2} \quad (1.4)$$

The time $t = 0$ corresponds to the end of motion toward the axis and the beginning of motion away from the axis. Thus $t > 0$ in problems on the expansion of a conductor and $t < 0$ for problems on compression.

In certain problems it is convenient to introduce the flux in the moving conductor

$$\Phi(a, t) = 2\pi \int_0^{(a^2+qt\pi^{-1})^{1/2}} rB(r, t)\,dr \quad (1.5)$$

or

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 18-25, September-October, 1973. Original article submitted March 23, 1973.

$$\psi(a, t) = 2\pi \int_{(a^2+qt\pi^{-1})^{1/2}}^{\infty} rB(r, t)\, dr \qquad (1.6)$$

instead of the field B (r, t).

If the field on the axis does not increase too rapidly (1.5) is used, and if the field and its derivatives vanish at infinity (1.6) is used. The flux Φ, ψ must satisfy the equation

$$\frac{d\Phi}{dt} = \frac{c^2}{2\sigma} r \frac{\partial B}{\partial r}, \qquad \frac{d\psi}{dt} = -\frac{c^2}{2\sigma} r \frac{\partial B}{\partial r} \qquad (1.7)$$

If $r\ \partial B/\partial r \to 0$ as $r \to \infty$ it follows from (1.7) that the flux is conserved in an unbounded conductor ($\Phi_{a\to\infty} = \Phi_0$). If the flux in the cavity is denoted by $\Phi_*(t)$ and the flux in the conductor by $\Psi_*(t)$ the conservation of total flux means that

$$\Phi_*(t) + \psi_*(t) = \Phi_0 \qquad (1.8)$$

In transforming to the self-similar variable

$$x = \pi r^2 (qt)^{-1} \qquad (1.9)$$

or to the dimensionless variables

$$\tau = t\ |q|\ (\pi r_0^2)^{-1}, \quad \rho = r^2 r_0^{-2} \qquad (1.10)$$

the dimensionless parameter

$$\mu = \sigma\, |q|\, c^{-2} \qquad (1.11)$$

equal to half the magnetic Reynolds number appears in all problems.

2. If the expansion of the cylindrical conductor begins at the axis two self-similar problems can be formulated: the conductor expands in a static field B_0; the conductor has a superconducting sheath and the flux in it is conserved. In both problems the solution depends on the single dimensionless variable x defined by (1.9); on the boundary of the conductor x = 1. For the expansion of a conducting cylinder in a static field

$$B(x) = B_0 \left(\int_0^1 \xi^{\mu-1} e^{-\mu\xi}\, d\xi\right)^{-1} \int_0^x \xi^{\mu-1} e^{-\mu\xi}\, d\xi$$

The field is zero on the axis of the conductor.

The cross section S of the conductor into which the field has penetrated is found from

$$B_0 S = 2\pi \int_0^{r_*} rB(r, t)\, dr = qt \int_0^1 B(x)\, dx$$

to be

$$S = \pi r_*^2 \left(\mu e^{\mu} \int_0^1 \xi^{\mu-1} e^{-\mu\xi}\, d\xi\right)^{-1} \qquad (2.1)$$

For large μ the integral in (2.1) can be approximated by Laplace's method [2], and for small μ by expanding the exponential in powers of $\mu\xi$

$$S \approx \pi r_*^2 (2\pi^{-1}\mu^{-1})^{1/2} \quad (\mu \gg 1),$$
$$S \approx \pi r_*^2 (1 - \mu) \quad (\mu \ll 1)$$

from which the depth of penetration of the field into the conductor is

$$\delta \approx r_* (2\pi\mu)^{-1/2} \quad (\mu \gg 1)$$
$$\delta \approx r_* (1 - \mu^{1/2}) \quad (\mu \ll 1)$$

In the problem of the expansion of the cylinder in a superconducting sheath the flux and field are respectively

$$\Phi(x) = \Phi_0 \left(\int_0^1 \xi^\mu e^{-\mu\xi} \, d\xi \right)^{-1} \int_0^x \xi^\mu e^{-\mu\xi} \, d\xi$$

$$B(x) = \frac{1}{2\pi r} \frac{\partial \Phi}{\partial r} = \Phi_0 \left(\pi r_*^2 \int_0^1 \xi^\mu e^{-\mu\xi} \, d\xi \right)^{-1} x^\mu e^{-\mu x}$$

The field vanishes on the axis and is maximum on the boundary of the conductor.

The cross section of the conductor occupied by the field is found from $B_* S = \Phi_0$, where

$$B_* = \Phi_0 \left(\pi r_*^2 \int_0^1 \xi^\mu e^{-\mu\xi} \, d\xi \right)^{-1} e^{-\mu}$$

is the field on the boundary of the conductor. It is easy to see that

$$S = \pi r_*^2 e^\mu \int_0^1 \xi^\mu e^{-\mu\xi} \, d\xi$$

and the methods described earlier give the estimates

$$\begin{aligned} &B_* \approx \Phi_0 (\pi r_*^2)^{-1} (2\pi^{-1}\mu)^{1/2} \\ &\delta \approx r_* \pi^{1/2} (8\mu)^{-1/2} \qquad (\mu \gg 1) \\ &B_* \approx \Phi_0 (\pi r_*^2)^{-1} (1 + 2^{-1}\mu) \\ &\delta \approx r_* (1 - (2^{-1}\mu)^{1/2}) \qquad (\mu \ll 1) \end{aligned}$$

3. If the expansion begins from the axis two self-similar problems in x can be formulated: at the beginning the field in the conductor was everywhere static and equal to B_0; at the beginning there was no field in the conductor and the flux Φ_0 was concentrated on the axis.

In the first problem the field in the conductor is

$$B(x) = B_0 - B_* \mu e^\mu \int_x^\infty \xi^{\mu-1} e^{-\mu\xi} \, d\xi$$

and the field in the cavity

$$B_* = B(1) = B_0 \left(1 + \mu e^\mu \int_1^\infty \xi^{\mu-1} e^{-\mu\xi} \, d\xi \right)^{-1} \tag{3.1}$$

is constant in spite of the motion of the conductor.

For large μ Laplace's method gives

$$B_* \approx B_0 (2\pi^{-1}\mu^{-1})^{1/2} \qquad (\mu \gg 1)$$

Making the substitution $\mu\xi = t$ in (3.1) leads to $\mu^{-\mu} [\Gamma(\mu) - \gamma(\mu, \mu)]$, and after substituting the expansions of the gamma function $\Gamma(z)$ and the incomplete gamma function $\gamma(a, z)$ for $|z| \ll 1$ [3] we obtain

$$B_* \approx B_0 (1 + \mu \ln \mu) \qquad (\mu \ll 1)$$

Direct calculations show that the decrease of the flux in the conductor is equal to the flux in the cavity, i.e.,

$$2\pi \int_{r_*}^\infty r (B_0 - B(r, t)) \, dr = \pi r_*^2 B_*$$

Thus the total flux is conserved in this case.

The cross section of the conductor from which the flux passed into the cavity is found from the equation $S(B_0 - B_*) = \pi r_*^2 B_*$. From this the thickness of the current layer on the surface of the expanding cavity is estimated as

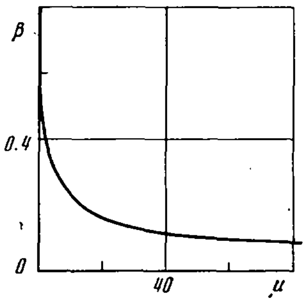

Fig. 1

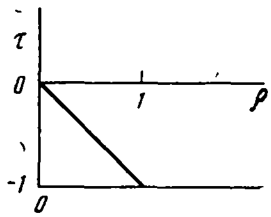

Fig. 2

$$\delta \approx 2^{-1} r_* \; B_* B_0^{-1} \approx r_* (2\pi\mu)^{-1/2} \quad (\mu \gg 1)$$

$$\delta \approx r_* (B_* (B_0 - B_*)^{-1})^{1/2} \approx r_* (-\mu \ln \mu)^{-1/2} \quad (\mu \ll 1)$$

The problem of the diffusion of the flux from an expanding cavity is discussed in [4]. It turns out that not only is the total flux conserved, but for $\nu = q/(2\pi r)$ both the flux in the cavity and the flux in the conductor are constant. Thus in the expansion of a cavity in a conductor the flux at the start is redistributed between the cavity and the conductor and then does not change.

Calculations for the flux in the cavity give

$$\Phi_* \approx \Phi_0 (1 - (2^{-1}\pi\mu^{-1})^{1/2}) \quad (\mu \gg 1)$$
$$\Phi_* \approx \Phi_0 \mu (1 + \mu \ln \mu) \quad (\mu \ll 1)$$

The cross section of the current layer at the edge of the cavity is

$$S = (\Phi_0 - \Phi_*) B_*^{-1} = \pi r_*^2 (\Phi_0 - \Phi_*) \Phi_*^{-1}$$

and its thickness is

$$\delta \approx r_* \pi^{1/2} (8\mu)^{-1/2} \quad (\mu \gg 1)$$
$$\delta \approx r_* \mu^{-1/2} \quad (\mu \ll 1)$$

4. Magnetic cumulation describes the problem of the compression of the flux in a cylindrical cavity. Introducing the flux in the conductor

$$\psi(r, t) = 2\pi \int_r^\infty r B(r, t)\, dr \tag{4.1}$$

and assuming that it can be written in the form

$$\psi(r, t) = (-t)^\beta \varphi(z),$$
$$z = \mu x = \sigma |q| c^{-2} \pi r^2 (qt)^{-1}$$

it is easy to show that $\varphi(z)$ must be a solution of the confluent hypergeometric equation with the parameters $-\beta$ and μ [5]

$$z\varphi'' + (\mu - z)\varphi' + \beta\varphi = 0 \tag{4.2}$$

satisfying the condition

$$\mu\varphi'(\mu) = \varphi(\mu) \tag{4.3}$$

on the boundary of the cavity $z = \mu$.

In deriving Eq. (4.2) it is assumed that the field vanishes at infinity. Therefore of the two linearly independent solutions of (4.2) we select the one whose asymptotic behavior at infinity is described by a power of z and discard the solution with the expotential asymptotic behavior, i.e.,

$$\varphi(z) = A\Psi(-\beta, \mu; z) \tag{4.4}$$

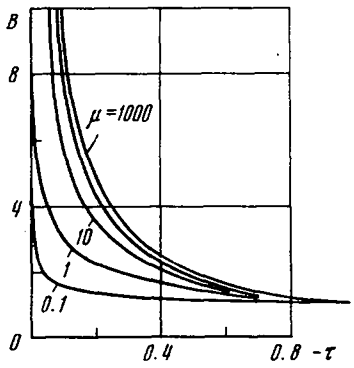

Fig. 3

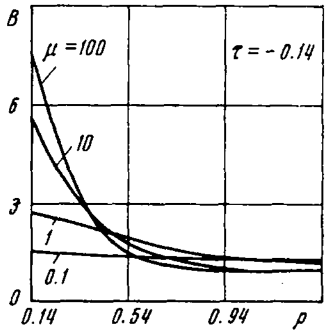

Fig. 4

where $\Psi(a, c, z)$ is given by the integral [5]

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{c-a-1} dt, \quad \mathrm{Re}\, a > 0 \tag{4.5}$$

The substitution of (4.4) into the boundary condition (4.3) and the transformations connected with the change of parameters in $\Psi(a, c, z)$ leads to an equation for β as a function of μ

$$(1-\beta)^2(2-\beta-\mu)\ \Psi(2-\beta,\ \mu;\ \mu) = [2(1-\beta)^2 + \beta(1-\beta-\mu)]\Psi(1-\beta,\ \mu;\ \mu) \tag{4.6}$$

Using the previously described technique to estimate the integral (4.5) we obtain the asymptotic forms

$$\begin{aligned} \beta &\approx (2\pi^{-1}\mu^{-1})^{1/2} \quad (\mu \gg 1) \\ \beta &\approx 1 - \mu \quad (\mu \ll 1) \end{aligned} \tag{4.7}$$

Equation (4.6) was solved on a Minsk-32 computer for arbitrary values of μ. The results of these calculations are shown in Fig. 1.

The solution shows that if at a time t_0 the radius of the cavity was $r_0 = (qt_0/\pi)^{1/2}$ and the flux in it was Φ_0, then at any other time t the flux in the cavity will be

$$\Phi_* = \Phi_0 (r_* r_0^{-1})^{2\beta} = \Phi_0 (t t_0^{-1})^\beta \tag{4.8}$$

and the field will be

$$B_* = \Phi_0 (\pi r_0^2)^{-1} (r_* r_0^{-1})^{2(\beta-1)}$$

Since $0 < \beta \le 1$, in compression ($r_* \to 0$) the flux passes completely into the conductor, although the field in the cavity increases indefinitely; i.e., leakage of the flux cannot limit the value of the field obtained in the compression of the flux in the cylindrical cavity.

It is clear from (4.7) and (4.8) that for large μ the flux in the cavity decreases slowly at first and only toward the end of the compression decreases sharply to zero. The time t^* for the flux to escape, and the critical size of the cavity r^* can be estimated from the condition

$$\left.\frac{d\Phi_*}{dt}\right|_{t=t^*} = \frac{\Phi_0}{t_0}$$

which combined with (4.8) gives

$$t^* = t_0 \beta^{1/(1-\beta)}, \quad r^* = r_0 \beta^{1/2(1-\beta)}$$

For large μ it follows from (4.7) that

$$\begin{aligned} t^* &\approx t_0 (2\pi^{-1}\mu^{-1})^{1/2} \\ r^* &\approx r_0 (2\pi^{-1}\mu^{-1})^{1/4} \quad (\mu \gg 1) \end{aligned} \tag{4.9}$$

A consideration of the compression of the flux in the cylindrical cavity shows that up to the time t^* the transport of the field by the conductor predominates, and at later stages of the compression the diffusion of the field into the conductor becomes controlling.

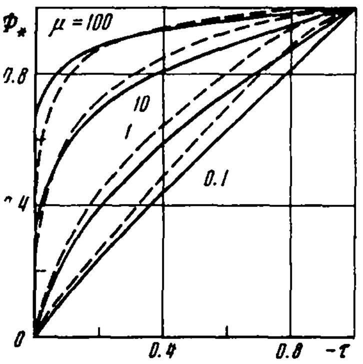

Fig. 5

The end of the compression corresponds to $z \to \infty$. At this instant the field and the flux in the conductor are

$$B_k(r) = \Phi_0(\pi r_0^2)^{-1}(\Psi(-\beta,\ \mu;\ \ \mu))^{-1}\beta\mu^{\beta}(rr_0^{-1})^{2(\beta-1)}$$
$$\Phi_k(r) = \Phi_0(\Psi(-\beta,\ \mu;\ \mu))^{-1}\mu^{\beta}(rr_0^{-1})^{2\beta} \tag{4.10}$$

After determining δ, the depth of penetration of the field into the conductor, by the relation $\Phi_k(\delta) = \Phi_0$ we can obtain from (4.10)

$$\delta = r_0\mu^{-1/2}(\Psi(-\beta,\ \mu;\ \mu))^{1/2\beta}$$

For large μ

$$\delta \approx r_0 \exp\left[-\frac{(1+\pi+c/2)}{2}\right](2\mu)^{-1/4} \qquad (\mu \gg 1) \tag{4.11}$$

Here C =0.577 . . . is Euler's constant [3]. It should be noted that the critical size of the cavity r^* (4.9) from which the flux passes rapidly into the conductor, and the depth of penetration of the field into the conductor δ at the end of the compression (4.11) are of the same order of magnitude for $\mu \gg 1$.

The power law (4.8) describing the decrease of the flux in the cavity for $\mu \gg 1$ agrees with the result cited in [6].

5. In order to find out under what circumstances self-similar solutions exist in magnetic cumulation problems, the problem of the compression of the field in a cylindrical cavity was solved numerically. It was assumed that at the start the field in the cavity and in the conductor was uniform and equal to B_0, and that the radius of the cavity was r_0. After transforming to the dimensionless variables τ and ρ defined by (1.10) the problem was reduced to the integration of the equation

$$\frac{\partial B}{\partial \tau} - \left(1 + \frac{1}{\mu}\right)\frac{\partial B}{\partial \rho} - \frac{\rho}{\mu}\frac{\partial^2 B}{\partial \rho^2} = 0 \tag{5.1}$$

in a cavity with a cut-off angle (Fig. 2) to calculate the field, or the equation

$$\frac{\partial \psi}{\partial \tau} - \frac{\partial \psi}{\partial \rho} - \frac{\rho}{\mu}\frac{\partial^2 \psi}{\partial \rho^2} = 0 \tag{5.2}$$

to calculate the flux ψ penetrating the conductor to a depth greater than r (4.1).

Since difference schemes for solving Eqs. (5.1) and (5.2) are stable for any value of $\Delta\tau/\Delta\rho$ [7], a square mesh $\Delta\tau = \Delta\rho$ was chosen for the calculation, and the problem was solved on a Minsk-32 computer. The mesh variables are related to the variables τ and ρ by the equations

$$I = M(\rho_\infty - \rho), \qquad J = M\tau$$

where 1/M is the mesh size. The coordinate ρ_∞ was chosen so that the time to close the cavity is small in comparison with the rise time of the skin layer at the depth ρ_∞. For $\mu \gg 1$ this condition gives $(\rho_\infty - 1) \gg (4\mu)^{-1}$.

The calculation determined the field and the flux in the conductor and the field in the cavity at various times for μ =0.1, 1, 10, 100, and 1000. The small computer memory did not permit calculations with a step smaller than 30^{-1}, so that the calculation is very crude near the instant the cavity is closed. For $\mu > 1$ the solution of Eq. (5.2) for the flux turned out to be more accurate, and for $\mu \leq 1$ the solution of Eq. (5.1) for the field was more accurate. The calculated values of the field in the cavity are shown in Fig. 3, and the field in the conductor at $\tau = -0.14$ is shown in Fig. 4.

A more accurate calculation of the flux in the cavity was performed by using the machine memory only for two neighboring lines in time, which permitted a reduction of the computing step to 500^{-1}. This organization of the calculation was possible only for large μ, since ρ_∞ is large when $\mu \leq 1$ and the calculation must extend a long way in the spatial variable. The open curves of Fig. 5 show the time dependence of the flux calculated in this way, and the solid curves are the self-similar solution (4.8). The solutions are joined at zero time. It is clear from a comparison of the two solutions that initially the compression of the flux decreases more slowly in the problem which is not self-similar than in the one which is; toward the end of the process the opposite is true. This occurs because at the beginning the field in the problem which is not self-similar is uniform in the conductor, and there is no flux leakage from the cavity. The initial phase of the compression of the field in this problem agrees with the solution of the compression of the flux by a perfect conductor. As the field in the cavity increases during its compression, the gradient of the field at the boundary increases more rapidly in the problem which is not self-similar than in the self-similar case, and according to (1.3) this leads to a more rapid passage of the flux from the cavity at the end of the compression.

In conclusion it should be noted that in all the problems discussed in Sec. 2 and 3 approximately the same asymptotic behavior was found for the depth of penetration of the field into a moving conductor for $\mu \gg 1$. This is natural since for large μ the transport of the field by the conductor is controlling. If the field decreases in the direction of motion, the depth of penetration of the field is $\pi/2$ times larger than when the velocity and the field gradient are oppositely directed. The sharp difference between the asymptotic behavior (4.11) of the solution for the compression of the field in the cavity and the solutions of the other problems for $\mu \gg 1$ is related to the growth of the field and its gradient at the boundary as the cavity is closed.

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